On maximally supersymmetric Yang-Mills theories

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Abstract

We consider ten-dimensional supersymmetric Yang-Mills theory (10D SUSY YM theory) and its dimensional reductions, in particular, BFSS and IKKT models. We formulate these theories using algebraic techniques based on application of differential graded Lie algebras and associative algebras as well as of more general objects, L_∞- and A_∞- algebras.

We show that using pure spinor formulation of 10D SUSY YM theory equations of motion and isotwistor formalism one can interpret these equations as Maurer-Cartan equations for some differential Lie algebra. This statement can be used to write BV action functional of 10D SUSY YM theory in Chern-Simons form. The differential Lie algebra we constructed

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is closely related to differential associative algebra $(\Omega, \bar{\partial})$ of $(0,k)$-forms on some supermanifold; the Lie algebra is tensor product of $(\Omega, \bar{\partial})$ and matrix algebra . We construct several other algebras that are quasiisomorphic to $(\Omega, \bar{\partial})$ and, therefore, also can be used to give BV formulation of 10D SUSY YM theory and its reductions. In particular, $(\Omega, \bar{\partial})$ is quasiisomorphic to the algebra (B, d), constructed by Berkovits. The algebras $(\Omega_0, \bar{\partial})$ and (B_0, d) obtained from $(\Omega, \bar{\partial})$ and (B, d) by means of reduction to a point can be used to give a BV-formulation of IKKT model.

We introduce associative algebra SYM as algebra where relations are defined as equations of motion of IKKT model and show that Koszul dual to the algebra (B_0, d) is quasiisomorphic to SYM.

1 Introduction.

The present paper begins a series of papers devoted to the analysis of the ten-dimensional supersymmetric Yang-Mills theory (10D SUSY YM theory), its dimensional reductions and its deformations having the same amount of supersymmetries. Theories of this kind are quite important. Dimensional reductions of 10D SUSY YM theory include the reduction to $(1+0)-$dimensional space first considered in [10] and used in [12] to describe supermembrane and in [4] (BFSS Matrix model) to propose mathematical formulation of M-theory , IKKT Matrix model [18] (the reduction to a point) proposed as a nonperturbative description of superstring , $N = 4$ SUSY YM theory in four-dimensional space. In some approximation dimensional reductions of 10D SUSY YM theory describe fluctuations of n coinciding flat D-branes; more precise description of D-branes is provided by supersymmetric deformation of reduced SUSY YM. We will analyze such deformations in [28]; this analysis can be used also to study Dirac-Born-Infeld (DBI) theory that can be considered as a SUSY deformation of SUSY YM where the terms containing derivations of field strength are neglected.(In our discussion of SUSY theories we always have in mind maximally supersymmetric gauge theories.)

We will use algebraic techniques based on application of differential graded Lie algebras and associative algebras as well as of more general objects, L_∞- and A_∞- algebras. In present paper we will give formulation of SUSY YM theories using these techniques. In the paper [27] the same techniques will be applied to more general gauge theories. The paper [26] will be devoted to calculation of some homology groups associated with the algebras considered in the present paper and in [27]. The consideration of supersymmetric deformations in [28] will be based on equivariant generalizations of results of [26].

Main results of the present paper and of the subsequent papers [27]-[28] can be proven rigorously. However in these papers sometimes we give only heuristic arguments, avoiding pretty complicated mathematical proofs. Rigorous proofs will appear in [25].

Let us describe some of constructions we are using.

If L is a differential graded Lie algebra we can write down the so called Maurer-Cartan equation

$$d\omega + \frac{1}{2}[\omega,\omega] = 0. \tag{1}$$

This equation appears in numerous mathematical problems. In particular, if $L = \sum_k \Omega^{0,k} \otimes Mat_n = \Omega \otimes Mat_n$ is the algebra of matrix-valued $(0,k)$ −forms on complex manifold X, then the differential $\overline{\partial} = d\bar{x}^k \cdot \partial/\partial\bar{x}^k$ specifies a structure of differential graded Lie algebra on L. (We can consider Ω and L also as differential graded associative algebras.) Solutions of the Maurer-Cartan equation (1)with $\omega \in \Omega^{0,1} \otimes Mat_n$ specify holomorphic structures on topologically trivial vector bundle. We will use this remark and isotwistor formalism [32], [13], [33] to write equations of motion of 10D SUSY YM as Maurer-Cartan equation for appropriate differential graded Lie algebra. There are various ways to write these equations of motion in such a form; such nonuniqueness is related to the fact that quasiisomorphic algebras have equivalent Maurer-Cartan equations. (A homomorphism of differential algebras is called a quasiisomorphism if it generates an isomorphism on homology.) The most convenient constructions are based on the manifold $\mathcal{Q} = \mathrm{SO}(10,\mathbb{R})/U(5)$ that can be interpreted as a

connected component of the isotropic Grassmannian or a space of pure spinors. (The relation of pure spinors to 10D SUSY YM was discovered and used in [16], [17], [7]).

Maurer-Cartan equations are closely related to Chern-Simons action functional; this general fact can be used to write down an action functional of Chern-Simons type that describes 10D SUSY YM in BV formalism .

Maurer-Cartan equations and their generalizations to $L_\infty-$algebras as well as corresponding Chern-Simons type action functionals play a very important role in our constructions. It seems that Chern-Simons type actions are ubiquitous; in some sense every action functional can be written in Chern-Simons form (see Appendix C.)

We are basing our consideration on geometric version of Batalin-Vilkovisky formalism [34], [35], [2]. The main geometric notion we are using is the notion of $Q-$manifold (supermanifold equipped with an odd vector field obeying $\{Q,Q\}=0$). An algebraic counterpart of this notion is the notion of $L_\infty-$algebra, that can be defined as a formal $Q-$ manifold. If a vector field Q vanishes at 0, then its Taylor expansion has the form

$$Q^a = Q^a_b x^b + Q^a_{bc} x^b x^c + \cdots$$

The coefficients of this expansion can be used to define algebraic operations $m_1, m_2, \ldots,$ that specify the structure of $L_\infty-$algebra if $\{Q,Q\}=0$.)

One can consider also non-commutative $Q-$manifolds (=$\mathbb{Z}_2$-graded associative algebras equipped with an odd derivation having square equal to zero= differential $\mathbb{Z}_2$-graded associative algebras) and $A_\infty-$algebras (= formal non-commutative $Q-$ manifolds.) We prefer to work not with differential graded Lie algebras and $L_\infty-$algebras but with differential graded associative algebras and $A_\infty-$algebras. There is a standard way to obtain a Lie algebra from associative algebras and $L_\infty-$algebra from $A_\infty-$algebra; at the very end we apply this construction. Considering associative and $A_\infty-$algebras we are studying gauge theories with all classical gauge groups at the same time.

We will give various formulations of 10D SUSY YM theory in terms of

differential associative algebras and $A_\infty-$algebras.

One can use, for example, the algebra $\Omega = \Sigma\Omega^{0,k}$ of $(0,k)-$forms on complex (super) manifold $\mathcal{R}$ with coordinates $\left(u^\alpha, \theta^\alpha, z^l\right)$ where u^α is a pure spinor (i.e. $u^\alpha \Gamma^a_{\alpha\beta} u^\beta = 0$) and

$$\left(u^\alpha, \theta^\alpha, z^l\right) \text{ is identified with } \left(\lambda u^\alpha, (\theta^\alpha - \varepsilon u^\alpha), (z^l - \varepsilon \Gamma^l_{\alpha\beta} u^\alpha \theta^\beta)\right)$$

(Here u^α, θ^α are ten-dimensional Weyl spinors, z^l are coordinates in ten-dimensional complex vector space, λ is a non-zero complex number, ε is an odd parameter.)

The algebra Ω will be regarded as an associative algebra equipped with differential $\bar{\partial}$. The equations of motion of 10D SUSY YM theory in BV-formalism coincide with Maurer-Cartan equations for differential Lie algebra corresponding to associative algebra $\Omega \otimes Mat_n$. The odd symplectic form of BV-formalism corresponds to inner product on the algebra Ω. (Recall, that in BV-formalism a classical system is described by a solution to the classical master equation $\{S, S\} = 0$ on odd symplectic manifold. Geometrically it corresponds to an odd vector field obeying $\{Q, Q\} = 0$ and $L_Q \sigma = 0$ where σ is a closed odd 2-form and L_Q stands for Lie derivative. In the language of $L_\infty-$algebras the form σ corresponds to an invariant odd inner product on $L_\infty-$algebra. This inner product should be non-degenerate on the homology of $L_\infty-$algebra.)

As we mentioned already we can use any differential graded algebra (or more generally A_∞-algebra) that is quasiisomorphic to Ω to describe super Yang-Mills theory. In particular, we can use the algebra B that consists of polynomial functions depending on pure spinor u^α, anticommuting spinor variable θ^α and 10 commuting variables $x^0, ..., x^9$. This algebra is equipped with a differential $d = u^\alpha D_\alpha$, where $D_\alpha = \frac{\partial}{\partial\theta^\alpha} - \Gamma^m_{\alpha\beta}\theta^\beta \frac{\partial}{\partial x^m}$. One can prove that differential algebra (B, d) is quasiisomorphic to the differential algebra $(\Omega, \bar{\partial})$(see Appendix A). The action functional of 10D super Yang-Mills theory again can be written as a Chern-Simons functional. The algebra (B, d) and corresponding action functional were introduced by Berkovits [7]. Notice that Berkovits gave a for-

mulation of superstring theory in terms of pure spinors; the SUSY YM theory can be analyzed as massless sector in the theory of open superstring [5],[6], [14], [15].

The paper is organized as follows. We start with reminder of main facts about ten-dimensional SUSY YM theory and about pure spinors (Sec. 2). In (Sec. 3) we use isotwistor formalism to write down equations of motion of 10D SUSY YM theory in the form of Maurer-Cartan equations for algebra Ω.This allows us to present 10D SUSY YM theory by means of Chern-Simons type action functional.

One can skip (Sec. 3) and go directly to (Sec. 4) containing a description of several quasiisomorphic differential algebras that can be used to analyze 10D SUSY YM theory and a construction of corresponding Chern-Simons type action functionals. The constructions of (Sec. 4) can be justified by means of results of (Sec. 3), however, there exist other ways to do this. In particular, one can justify these constructions using the language of $A_\infty-$algebras; we sketch such a derivation in (Sec. 5). Another way to justify our statements will be described in [27]. Appendices A and B contain some omitted proofs. The proofs of Appendix B are based on some general statements, that have also other interesting applications. Appendix C is devoted to the exposition of some basic facts about L_∞- and A_∞-algebras, we follow mostly [2], [21], [19] . (This appendix, included for readers convenience, is a part of paper [29] containing review of the theory of L_∞- and A_∞-algebras and some new results.)

2 Maximally supersymmetric gauge theories.

Let us start with consideration of ten-dimensional supersymmetric gauge theory and its reductions. In components this theory is described by the action functional

$$S_{SYM}(A_m, \chi^a) = -\frac{1}{4}\mathrm{Tr}\, F_{mn}F^{mn} + \frac{1}{2}\mathrm{Tr}\, \chi^\alpha \Gamma^m_{\alpha\beta}[\mathcal{D}_m, \chi^\beta] \tag{2}$$

where $A_m(x)$ and $\chi^\alpha(x)$ are matrix valued functions on $\mathbb{R}^{10}$, $m = 0, 1, \dots, 9$ is a vector index, $\alpha = 1, \dots, 16$ is a spinor index, $\mathcal{D}_m = \frac{\partial}{\partial x_m} + A_m(x)$ are covariant derivatives, $F_{nm} = \partial_m A_n - \partial_n A_m + [A_m, A_n]$ stands for field strength. Notice, that in ten-dimensional space left spinors χ^α and right spinors φ_α are dual to each other ; the ten-dimensional Γ -matrix takes the form

$$\Gamma_m = \begin{pmatrix} 0 & \Gamma_m^{\alpha\beta} \\ (\Gamma_m)_{\alpha\beta} & 0 \end{pmatrix}$$

The fermions χ^α are considered as anticommuting variables. We consider our action functional S as complex analytic functional of fields. To quantize our theory we should integrate e^{-S} over a real slice in the complex space of fields. If we introduce Minkowski metric on $\mathbb{R}^{10}$ we can single out the real slice imposing Hermiticity condition on A_m and Majorana condition on χ^α . With this choice the real slice is invariant with respect to symmetries of the theory. It is important to notice, however, that deforming the slice we don't change the value of the functional integral. Therefore, there is no necessity to respect the symmetries of the theory choosing the real slice. This remark is important, for example, in the case of Euclidean metric on $\mathbb{R}^{10}$, when the notion of Majorana spinor is not defined. We will be interested only in symmetries of the theory therefore we will always work with complex analytic action functional; we have no necessity to discuss the choice of real slice.

It will be convenient for us to work with holomorphic fields $A_m(x), \chi^\alpha(x)$ defined on complex space $V = \mathbb{C}^{10}$ and taking values in $V^* \otimes Mat_n$ and $\Pi S \otimes Mat_n$ correspondingly. (Here Mat_n stands for the algebra of complex matrices, S denotes left spinor representation and Π means parity reversion).

All our considerations will be local with respect to $x \in V$. In other words, our fields are polynomials or formal power series with respect to x. This means, in particular, that action functionals we consider are formal expressions (they contain an ill-defined integral over x)

The group of symmetries of action (2) includes the two-sheeted covering $Spin(10, \mathbb{C})$ of group $SO(10, \mathbb{C})$.

The action (2) is also invariant under gauge transformations parametrized by an endomorphism ϕ:

$$V_\phi(A_m) = [\mathcal{D}_m, \phi]\,, \qquad V_\phi(\chi^\alpha) = [\chi^\alpha, \phi]\,.$$

The action functional (2) is invariant under the supersymmetry transformations

$$\begin{aligned} \delta_\epsilon(A_m) &= \epsilon^\alpha (\Gamma_m)_{\alpha\beta}\chi^\beta\,, \\ \delta_\epsilon(\chi^\alpha) &= \frac{1}{2}(\Gamma^{mn})^\alpha{}_\beta \epsilon^\beta F_{mn} \end{aligned}$$

as well as under trivial supersymmetry transformations

$$\tilde{\delta}_\epsilon(A_m) = 0\,, \quad \tilde{\delta}_\epsilon(\chi^\alpha) = \epsilon^\alpha\,.$$

Each of the last two transformations is parametrized by ϵ^α - a constant Weyl spinor, i.e. a spinor proportional to the unit endomorphism.

The equations of motion corresponding to (2) (but not the action functional itself) can be written in explicitly supersymmetric form in $(10|16)$ -dimensional superspace $\mathbb{C}^{10|16}$. To do this we define a connection on this space as a collection of covariant derivatives $D_m = \partial_m + A_m(x,\theta)$, $D_\alpha = D_\alpha + A_\alpha(x,\theta)$ where $\partial_m = \frac{\partial}{\partial x^m}$, $D_\alpha = \frac{\partial}{\partial \theta^\alpha} - \Gamma_{\alpha\beta}{}^m \theta^\beta \partial_m$, A_m and A_α are holomorphic matrix functions of ten-dimensional vector x^m and Weyl spinor θ^α (we consider θ^α as anticommuting variables). The curvature of the connection $D = (D_m, D_\alpha)$ has components

$$\begin{aligned} &F_{\alpha\beta} = \{\mathcal{D}_\alpha, \mathcal{D}_\beta\} + 2\Gamma^m_{\alpha\beta}\mathcal{D}_m\,, \\ &F_{\alpha m} = [\mathcal{D}_\alpha, \mathcal{D}_m]\,, \qquad F_{mn} = [\mathcal{D}_m, \mathcal{D}_n]\,. \end{aligned} \tag{3}$$

We will say that a connection D is a gauge superfield if it satisfies the constraint

$$F_{\alpha\beta} = 0 \tag{4}$$

It is easy to check that the set of gauge superfields is invariant with respect to supergauge transformations

$$A_m \mapsto g^{-1}A_m g + g^{-1}\mathcal{D}_m g\,, \quad A_\alpha \mapsto g^{-1}A_\alpha g + g^{-1}\mathcal{D}_\alpha g\,. \tag{5}$$

and supersymmetry transformations

$$\delta_\gamma(\mathcal{D}_m) = [\mathcal{D}_m, \tilde{D}_\gamma]\,, \quad \delta_\gamma(\mathcal{D}_\alpha) = \{\mathcal{D}_\alpha, \tilde{D}_\gamma\}\,. \tag{6}$$

where

$$\tilde{D}_\alpha = \frac{\partial}{\partial\theta^\alpha} + \Gamma^m_{\alpha\beta}\theta^\beta\partial_m\,. \tag{7}$$

One can identify the moduli space of gauge superfields (the set of solutions of (4) factorized with respect to supergauge transformations) with the moduli space of solutions to the equations

$$\begin{aligned} &\Gamma^m_{\alpha\beta}\mathcal{D}_m\chi^\beta = 0 \\ &\mathcal{D}_m F^{mn} = \frac{1}{2}\Gamma^n_{\alpha\beta}\{\chi^\alpha, \chi^\beta\} \end{aligned} \tag{8}$$

(space of solutions of the equations of motion corresponding to the action functional (2) up to gauge transformations.) To prove this statement one can notice that the gauge condition

$$\theta^\alpha A_\alpha = 0 \tag{9}$$

restricts the group of supergauge transformation to the group of gauge transformations. All components of gauge superfield $(\mathcal{A}_m, \mathcal{A}_\alpha)$ obeying gauge condition (9) can be restored by means of some recurrent process [1] from the zero components of superfields A_m and $\chi^\alpha = (\Gamma^m)^{\alpha\beta}F_{\beta m}$. These components are identified with corresponding fields in the component formalism; they obey the equation (8).

Notice that using (4) one can express $A_m(x,\theta)$ in terms of $A_\alpha(x,\theta)$. This remark permits us to write the equation of motion in superfield formalism as a condition

$$\{D(u), D(u)\} = 0,$$

where $D(u) = u^\alpha D_\alpha = u^\alpha(D_\alpha + A_\alpha)$ and u^α is an arbitrary none-zero spinor obeying $u^\alpha\Gamma_{\alpha\beta}{}^m u^\beta = 0$ (in other words u^α are ten-dimensional pure spinor). This pure spinor formulation of equations of motion [16], [17] can be used to apply the ideas of isotwistor formalism [16], [13], [33]. The manifold of pure spinors $\tilde{\mathcal{Q}}$ and corresponding projective manifold $\mathcal{Q}$ play the main role in this

formulation. (In $\mathcal{Q}$ we identify spinors u^α and λu^α. In other words $\mathcal{Q}$ is obtained from $\tilde{\mathcal{Q}}$ by means of factorization with respect to vector field $E = u^\alpha \frac{\partial}{\partial u^\alpha}$). It is clear that $\mathcal{Q}$ and $\tilde{\mathcal{Q}}$ are complex manifolds, $dim_{\mathbb{C}}\mathcal{Q} = 10$, $dim_{\mathbb{C}}\tilde{\mathcal{Q}} = 11$. The group $SO(10,\mathbb{R})$ acts as on $\mathcal{Q}$ transitively with stable subgroup $U(5)$. This means that we can identify $\mathcal{Q}$ with $SO(10,\mathbb{R})/U(5)$. The manifold $\tilde{\mathcal{Q}}$ can be considered as a total space of $\mathbb{C}^*$ bundle α over $\mathcal{Q}$ or as a homogeneous space $Spin(10,\mathbb{R})\times\mathbb{C}^*/\tilde{U}(5)$.(Here $\mathbb{C}^*$ stands for the multiplicative group of non-zero complex numbers and $\tilde{U}(5)$ is a two-sheeted cover of $U(5)$).

Sometimes it is convenient to use instead of $\widetilde{\mathcal{Q}}$ the manifold $\widehat{\mathcal{Q}} = Spin(10,\mathbb{R})\times \mathbb{C}/\widetilde{U}(5)$ (the total space of $\mathbb{C}$-bundle $\widehat{\alpha}$ over $\mathcal{Q}$).

The complex group $Spin(10,\mathbb{C})$ also acts transitively on $\mathcal{Q}$; corresponding stable subgroup P is a parabolic subgroup. To describe the Lie algebra $\mathfrak{p}$ of P we notice that the Lie algebra $\mathfrak{so}(10,\mathbb{C})$ of $SO(10,\mathbb{C})$ can be identified with $\Lambda^2(V)$ (with the space of antisymmetric tensors ρ_{ab} where $a,b = 0,\dots,9$). The vector representation V of $SO(10,\mathbb{C})$ restricted to the group $GL(5,\mathbb{C}) \subset SO(10,\mathbb{C})$ is equivalent to the direct sum $W \oplus W^*$ of vector and covector representations of $GL(5,\mathbb{C})$. The Lie algebra of $SO(10,\mathbb{C})$ as vector space can be decomposed as $\Lambda^2(W) \oplus \mathfrak{p}$ where $\mathfrak{p} = (W \otimes W^*) \oplus \Lambda^2(W^*)$ is the Lie subalgebra of $\mathfrak{p}$. Using the language of generators we can say that the Lie algebra $\mathfrak{so}(10,\mathbb{C})$ is generated by skew-symmetric tensors m_{ab}, n^{ab} and by k_a^b where $a,b = 1,\dots,5$. The subalgebra $\mathfrak{p}$ is generated by k_a^b and n^{ab}. Corresponding commutation relations are

$$[m,m'] = [n,n'] = 0 \tag{10}$$

$$[m,n]_a^b = m_{ac}n^{cb} \tag{11}$$

$$[m,k]_{ab} = m_{ac}k_b^c + m_{cb}k_a^c \tag{12}$$

$$[n,k]_{ab} = n^{ac}k_c^b + n^{cb}k_c^a \tag{13}$$

3 Isotwistor formalism.

To apply the ideas of isotwistor formalism to equations of motion in the form (4) we need some information about holomorphic vector bundles on the manifold $\mathcal{Q}$ of pure spinors. We will assume that every topologically trivial holomorphic vector bundle over $\mathcal{Q}$ is holomorphically trivial. This statement can be proved for semistable bundles [1]; holomorphic triviality of small deformation of holomorphically trivial bundle immediately follows from $H^{0,1}(\mathcal{Q}) = 0$.

Let as introduce a supermanifold $\tilde{\mathcal{R}}$ as a direct product of $\tilde{\mathcal{Q}}$, V and ΠS. Here V stands for the space of vector representation of $SO(10,\mathbb{C})$ and S for the space of its spinor representation. (We are working always with left spinors.) The manifold $\tilde{\mathcal{R}}$ can be interpreted also as a quotient $G \times \mathbb{C}^*/P$ where G stands for complex super Poincare group and $P \subset Spin(10,\mathbb{C})$ is embedded in natural way into $G \times \mathbb{C}^*$. A point of $\tilde{\mathcal{R}}$ can be represented as a triple $(u^\alpha, x^m, \theta^\alpha)$ where $u^\alpha$is a pure spinor , $x^m \in V$, $\theta^\alpha \in \Pi S$. We define an odd vector field on $\tilde{\mathcal{R}}$ by the formula

$$D = u^\alpha D_\alpha = u^\alpha \left(\frac{\partial}{\partial \theta^\alpha} + \Gamma_{\alpha\beta}{}^m \theta^\beta \frac{\partial}{\partial x^m}\right)$$

It satisfies $D^2 = 0$. A complex manifold $\mathcal{R}$ will be defined as a quotient of $\tilde{\mathcal{R}}$ with respect to vector fields D and

$$E = u^\alpha \frac{\partial}{\partial u^\alpha}$$

The vector fields D and E commute with the action of super Poincare group G. This means that $\mathcal{R}$ can be considered as a quotient of G with respect to a subgroup generated by a semidirect product of P and a subgroup of dimension $(0|1)$.(If S is represented as a fermionic Fock space the generator of this subgroup corresponds to the vacuum.) It will be convenient to consider also the manifold $\mathcal{R}'$ obtained from $\tilde{\mathcal{R}}$ by means of factorization with respect to vector field E. This manifold is intermediate between $\tilde{\mathcal{R}}$ and $\mathcal{R}$; to obtain $\mathcal{R}$ from $\mathcal{R}'$ we factorize with respect to D.

[1] A. Tyurin, private communication

The manifold $\mathcal{R}$ is homogeneous with respect to the action of super-Poincare group.

Let us consider now the algebra Ω of $(0,k)$-forms on the (super) manifold $\mathcal{R}$. This is a supercommutative associative algebra equipped with differential $\bar{\partial}$. Introduce also

$$\Omega \otimes Mat_n \tag{14}$$

(the algebra of matrix-valued $(0,k)$-forms). Considering (14) as a differential Lie algebra we can write down the corresponding Maurer-Cartan equation $\bar{\partial}\omega + \frac{1}{2}\{\omega,\omega\} = 0$. We will show that in the case when ω is a $(0,1)$-form this equation is equivalent to (4). First of all, we notice that $\Omega \otimes Mat_n$ is a subalgebra of the algebra of $(0,k)$ -forms on $\tilde{\mathcal{R}}$. More precisely, vector fields D, E and their complex conjugate fields $\bar{D}, \bar{E}$ determine operators $L_D, L_{\bar{D}}, i_D, i_{\bar{D}}, L_E, L_{\bar{E}}, i_E, i_{\bar{E}}$ (Lie derivatives and contractions). A form ω on $\tilde{\mathcal{R}}$ descends to $\mathcal{R}$ if

$$\begin{aligned} L_D\omega = L_{\bar{D}}\omega = i_D\omega = i_{\bar{D}}\omega = 0 \\ L_E\omega = L_{\bar{E}}\omega = i_E\omega = i_{\bar{E}}\omega = 0 \end{aligned} \tag{15}$$

(We work with $(0,k)$-forms hence $i_D\omega, i_E\omega$ always vanish.)

We will consider a matrix valued $(0,1)$-form ω on $\tilde{\mathcal{R}}$ that obeys (15) and satisfies Maurer-Cartan equations. It descends to $\mathcal{R}$ and to $\mathcal{R}'$. On $\mathcal{R}'$ it defines a topologically trivial holomorphic vector bundle; such a bundle is also holomorphically trivial. (One can derive this statement from corresponding statement for manifold $\mathcal{Q}$.) This means that there exists a matrix-valued function g on $\tilde{\mathcal{R}}$ that obeys $\omega = g^{-1}\bar{\partial}g$ and descends to $\mathcal{R}'$ (satisfies $Eg = \bar{E}g = 0$). The operator gDg^{-1} can be written in the form

$$gDg^{-1} = D + u^{\alpha}A_{\alpha} = u^{\alpha}\left(D_{\alpha} + A_{\alpha}\right).$$

It follows from $\left\{gDg^{-1}, gDg^{-1}\right\} = 0$ that A_{α} obeys the Yang-Mills equation of motion. From the other side it follows from (4) that for every solution of YM equations of motion $A_{\alpha}(x,\theta)$ there exists such a matrix-valued function g on $\tilde{\mathcal{R}}$ that

$$gDg^{-1} = D(u) = u^{\alpha}\left(D_{\alpha} + A_{\alpha}\right)$$

moreover, function g can be chosen in such a way that

$$g^{-1} \cdot \overline{D} \cdot g = \overline{D}, \quad g^{-1} \cdot E \cdot g = E, \quad g^{-1} \cdot \overline{E} \cdot g = \overline{E},$$

or, in other words,

$$\overline{D} g = 0, \quad E g = 0, \quad \overline{E} g = 0.$$

Then the form $\omega = g^{-1} \overline{\partial} g$ obeys Maurer-Cartan equations and the conditions (15).

4 Various formulations of 10D SUSY YM. Chern-Simons action functionals.

We have shown how one can express the SUSY YM equations of motion in the form of Maurer-Cartan equations of motion for the algebra $\Omega \otimes Mat_n$. Recall that $\Omega = \sum \Omega^{0,k}$ stands for the algebra of $(0,k)$-forms on complex supermanifold $\mathcal{R}$. (We will use the notation $\Omega(M)$ for the algebra of $(0,k)$-forms on a manifold M; hence $\Omega = \Omega(\mathcal{R})$.) As we mentioned in the introduction many important theories can be obtained by means of dimensional reduction of 10D SUSY YM; in particular, IKKT Matrix model is obtained by means of reduction to a point. Of course, all of these theories can be obtained also by means of dimensional reduction of action functional (2). For example, IKKT Matrix model can be described by means of the algebra Ω_0 (dimensional reduction of Ω) . More precisely, the algebra Ω_0 is defined as $(\Omega(\mathcal{R}_0), \overline{\partial})$ where $\mathcal{R}_0$ is obtained from $\widetilde{\mathcal{R}}_0 = \widetilde{\mathcal{Q}} \times \Pi S$ by means of factorization with respect to vector fields $E = u^\alpha \frac{\partial}{\partial u^\alpha} + \theta^\alpha \frac{\partial}{\partial \theta^\alpha}$ and $d = u^\alpha \frac{\partial}{\partial \theta^\alpha}$.

We will discuss now BV-formalism of reduced theory in terms of algebra Ω_0; the generalization to complete 10D SUSY YM theory is straightforward.

It will be essential later that $\mathcal{R}_0$ is a split supermanifold (a supermanifold that can be obtained from a vector bundle by means of parity reversion on fibers). Moreover, the manifold $\mathcal{Q}$ (underlying manifold for $\mathcal{R}_0$) is a complex

manifold equipped with a transitive holomorphic action of the group $SO(10, \mathbb{R})$ with a stabilizer $U(5)$. $\mathcal{R}_0$ can be obtained from a homogeneous vector bundle corresponding [2] to two-valued representation T of $U(5)$. Here T is $(\Lambda^2(W) + \Lambda^4(W)) \otimes det^{-\frac{1}{2}}(W)$ where W stands for vector representation of $U(5)$.

Let us write down the BV action functional for reduced 10D SUSY YM theory in terms of Ω_0 as a generalized Chern-Simons action functional.

Namely, the action functional is defined on the algebra $\Omega_0 \otimes Mat_n$ by the formula

$$f(\omega) = Tr\left(\frac{1}{2}\omega\bar{\partial}\omega + \frac{2}{3}\omega\left[\omega, \omega\right]\right) \tag{16}$$

where the functional Tr does not vanish only on the forms of degree 3.

The construction of functional Tr involves integration. In the integration theory on supermanifolds one should use so called integral forms [8] instead of differential forms. An integral k -form can be defined as an expression

$$\psi = C^{A_1 \ldots A_k}(z)\rho(z)$$

where z^A are (even and odd) coordinates on a supermanifold, $\rho(z)$ stands for volume element, $C^{A_1 \ldots A_k}$ is a (super) antisymmetric tensor of rank k. In mathematical terms ρ is a section of Berezinian line bundle Ber (a bundle having (super)Jacobians as transition functions) and coefficients $C^{A_1 \ldots A_k}$ are sections of $\Lambda^k(T)$ where T stands for tangent bundle and Λ^k is an exterior power in the sense of superalgebra [3].

Suppose we are given a super-manifold of dimension (a, b), an integral form ψ of degree k and immersed super-manifold X of dimension $(a - k, b)$. Then there is a well defined operation of integration of the form ψ over submanifold X. In other words immersed submanifolds of right dimension define functionals

[2] Recall that for a representation of H on vector space V we can construct a homogeneous vector bundle G/H having $(G \times V)/H$ as the total space and V as a fiber.

[3] Recall that for $\mathbb{Z}_2$-graded space $T = T_0 + T_1$ the (super)exterior algebra $\Lambda^\bullet(T)$ is $\Lambda^\bullet(T_0) \otimes S^\bullet(T_1)$ where $S^\bullet$ stands for symmetric algebra

on integral forms. There is a different method to define functionals on the space Ω^{-k}_{int} of integral k -forms. Take a differential form λ of degree k. After making contractions of vector and covector indexes one gets a section $<\psi, \lambda>$ of Ber which can be integrated over the manifold. Hence we obtain a pairing between integral and differential forms. One can introduce an analog of de Rham differential on the space of integral forms and to prove an analog on Stokes' formula. The differential is compatible with the pairing of differential and integral forms.

On a complex manifold a real line bundle Ber is a tensor product $Ber_{\mathbb{C}} \otimes \overline{Ber}_{\mathbb{C}}$, where $Ber_{\mathbb{C}}$ is a holomorphic line bundle, called a holomorphic Berezinian, the bar symbol means a complex conjugation. The symbol $\otimes$ in context of vector bundles will always mean their fiberwise tensor product.

A complexification of the tangent vector bundle T of a complex manifold splits into a sum $T_{\mathbb{C}} + \overline{T}_{\mathbb{C}}$, where $T_{\mathbb{C}}$ is a holomorphic and $\overline{T}_{\mathbb{C}}$ is an antiholomorphic component. This implies that

$$\Omega^{-k}_{int} = Ber \otimes \Lambda^k(T) = \bigoplus_{i+j=k} Ber_{\mathbb{C}} \otimes \Lambda^i(T_{\mathbb{C}}) \otimes \overline{Ber}_{\mathbb{C}} \otimes \Lambda^j(\overline{T}_{\mathbb{C}})$$

In other words, on complex manifold we can consider not only differential (i,j)-forms, but also integral (i,j)-forms.

The integral form we about to construct is a $(0,3)$-form, i.e. it lives in $Ber_{\mathbb{C}} \otimes \overline{Ber}_{\mathbb{C}} \otimes \Lambda^3(\overline{T}_{\mathbb{C}})$. In the case at hand we are working with a split supermanifold $\mathcal{R}_0$. The space $(\Lambda^3(W) \oplus \Lambda^1(W)) \otimes det^{-1}(W) \otimes \overline{\Lambda^3(W) \otimes det^{-1}(W)}$ contains a nontrivial $U(5)$ invariant element, defined up to a constant. If one takes a seventh exterior power of it, one gets an element

$$tr \in \Lambda^7(\Lambda^3(W) \oplus \Lambda^1(W)) \otimes det^{-7}(W) \otimes \overline{\Lambda^7(\Lambda^3(W)) \otimes det^{-7}(W)} \tag{17}$$

We will use this element to construct an $SO(10)$-invariant integral form. Smooth sections of holomorphic Berezinian $Ber_{\mathbb{C}}$ can be identified with sections of a homogeneous vector bundle over $\mathcal{Q}$ corresponding to the following representation of $U(5)$

$$\mathcal{B} = A^{\bullet} \otimes \overline{A}^{\bullet} \otimes det^{-\frac{7}{2}}(W) \tag{18}$$

where

$$A^\bullet = \Lambda^\bullet[(\Lambda^3(W) \oplus \Lambda^1(W)) \otimes det^{-\frac{1}{2}}(W)]$$

The smooth sections of vector bundle over $\mathcal{Q}$ that corresponds to $A^\bullet \otimes \overline{A}^\bullet$ can be identified with the ring of smooth superfunctions on the manifold $\mathcal{R}_0$.

Sections of vector bundles over $\mathcal{R}_0$ are modules over the ring of superfunctions on $\mathcal{R}_0$. Sections of homogeneous vector bundles over $\mathcal{R}_0$ can be interpreted as sections of some homogeneous vector bundles over $\mathcal{Q}$, which in turn are induced from $U(5)$ representations. These representations are modules over $A^\bullet \otimes \overline{A}^\bullet$-the ring which is the substitute in the representation theory for the ring of superfunctions on $\mathcal{R}_0$. In the following if representations X and Y are modules over $A^\bullet \otimes \overline{A}^\bullet$, we denote their tensor product over $A^\bullet \otimes \overline{A}^\bullet$ as $X \underset{A^\bullet \otimes \overline{A}^\bullet}{\otimes} Y$. Such tensor product of representations corresponds to the tensor product of vector bundles.

The line bundle $Ber_{\mathbb{C}}$ over $\mathcal{R}_0$ is a pullback of a homogeneous line bundle L over $\mathcal{Q}$, that corresponds to $U(5)$ representation $det^{-\frac{7}{2}}(W)$ (this is a local computation).

The third exterior power of tangent bundle to $\mathcal{R}_0$ corresponds to representation

$$\Lambda^3(T) = \left(\bigoplus_{i=0}^{3} \Lambda^i(\Lambda^2(W)) \otimes S^{3-i}(\Lambda^2(W) \oplus \Lambda^4(W)) \otimes det^{-\frac{3-i}{2}}(W)\right) \otimes A^\bullet \otimes \overline{A}^\bullet$$

Smooth integral $(0,3)$-forms on manifold $\mathcal{R}_0$ can be considered as sections of a vector bundle induced from representation

$$\mathcal{B} \underset{A^\bullet \otimes \overline{A}^\bullet}{\otimes} \overline{\mathcal{B} \underset{A^\bullet \otimes \overline{A}^\bullet}{\otimes} \Lambda^3(T_{\mathcal{R}})}.$$

It is clear that $\overline{\Lambda^7(\Lambda^3(W)) \otimes det^{-7}(W)}$ can be identified with

$$\begin{aligned}
&\overline{\Lambda^3(\Lambda^2(W)) \otimes det^{-\frac{8}{2}}(W)} \subset \\
&\subset \overline{\Lambda^3(\Lambda^2(W)) \otimes \Lambda^{15}[(\Lambda^3(W) \oplus \Lambda^1(W)) \otimes det^{-\frac{1}{2}}(W)] \otimes det^{-\frac{7}{2}}(W)} \subset \\
&\subset \overline{\mathcal{B} \underset{A^\bullet \otimes \overline{A}^\bullet}{\otimes} \Lambda^3(T_{\mathcal{R}})}.
\end{aligned}$$

We see that tr introduced in (17) defines a $U(5)$-invariant element of $\mathcal{B} \underset{A^\bullet \otimes \overline{A}^\bullet}{\otimes}$ $\overline{\mathcal{B} \underset{A^\bullet \otimes \overline{A}^\bullet}{\otimes} \Lambda^3(T_{\mathcal{R}})}$. The corresponding $SO(10)$- invariant section is the integral form we are looking for. We denote this section by the same symbol tr. It is easy to prove that the element $tr \in Ber_{\mathbb{C}} \otimes \Lambda^3(\overline{T}_{\mathbb{C}}) \otimes \overline{Ber}_{\mathbb{C}} \subset \Omega^{-3}_{int}$ is a d closed integral form. It can be used to define a functional of the space of differential $(0,3)$-forms on manifold $\mathcal{R}_0$ by the formula

$$Tr(a) = \int_{\mathcal{R}_0} < tr, a >$$

Using the functional Tr we define an odd 2-form on Ω_0 :

$$\sigma(a,b) = Tr(a \wedge b)$$

This 2-form defines an odd $\overline{\partial}$-invariant presymplectic structure on $\Omega_0 \otimes Mat_n$. The functional (16) obeys master equation with respect to this presymplectic structure. (More precisely, it descends to corresponding odd symplectic manifold and satisfies master equation there.) This follows from the fact that the equations of motion corresponding to (16) are Maurer-Cartan equations coming from an odd vector field having square equal to zero.

To write down an odd presymplectic structure on $\Omega_0 \otimes Mat_n$ and BV action functional for 10D SUSY YM theory we use the same formulas as in reduced case adding everywhere integration with respect to spatial coordinates.

Notice, that the formulation of SUSY YM and its reductions in terms of the algebra of $(0,k)$-forms is similar to the formulation of B-model on Calabi-Yau manifold in terms of holomorphic Chern-Simons action functional. The algebra of B-model is the same, but the odd symplectic structure is different. One can say that we formulated SUSY YM as a generalized B-model on a supermanifold.

Let us describe several algebras that are quasiisomorphic to Ω and Ω_0 and, therefore can be used to analyze 10D SUSY YM and its reduction to a point.

The algebra B_0 is defined as an algebra of polynomial functions of pure spinor u^α and anticommuting spinor θ^α. It is equipped with differential

$$d = u^\alpha \frac{\partial}{\partial \theta^\alpha}.$$

One can say that B_0 is a tensor product of the algebra $F(\widetilde{\mathcal{Q}})$ of polynomial functions on the manifold $\widetilde{\mathcal{Q}}$ and exterior algebra $\Lambda(S^*) = F(\Pi S)$. The algebra B_0 can be obtained by means of reduction to a point from the algebra $B = B_0 \otimes F(V)$ equipped with differential

$$d = u^\alpha(\frac{\partial}{\partial\theta^\alpha} + \Gamma^m_{\alpha\beta}\theta^\beta\frac{\partial}{\partial x^m})$$

where $F(V)$ stands for the space of functions depending of $x^1, ..., x^m$.

One can prove that the algebra B is quasiisomorphic to Ω and B_0 is quasiisomorphic to Ω_0 (see Appendix A.) This statement permits us to formulate 10D SUSY YM theory in terms of algebra B and its reduction to a point (IKKT model) in terms of B_0. The action functional again has Chern-Simons form

$$f\left(\omega\right) = Tr\left(\frac{1}{2}\omega d\omega + \frac{2}{3}\omega\left[\omega,\omega\right]\right) \tag{19}$$

where Tr stands for the functional on B_0 that is determined by the following properties. Let us define the space $B_0^{5,3}$ as a subspace of B_0, generated by monomials $u^{\alpha_1}\dots u^{\alpha_3}\theta^{\beta_1}\dots\theta^{\beta_5}$. It contains a unique (up to a factor) $Spin(10)$-invariant non-zero vector λ. Denote a $Spin(10)$- invariant projection on λ by p. Then by definition

$$Tr(\omega)\lambda = p(\omega).$$

The functional Tr is unique up to multiplication on non-zero constant. (We considered here the action functional of IKKT model; to obtain the action functional of full 10D SUSY YM theory one should work with algebra B and include integration over $\mathbb{R}^{10}$ in the definition of action functional.)

One more description of IKKT Matrix model can be obtained from isotwistor formulation [33] of $N = 3$ four-dimensional SUSY YM theory (that is equivalent to its $N = 4$ counterpart after reduction to a point). This description is based on the following algebra $R_{N=3}$ described below. The algebra $R_{N=3}$ is quasiisomorphic to Ω_0 and B_0.

Let $k[p_1, \dots, p_3, u^1, \dots, u^3, z_1, z_2, w_1, w_2]$ be a polynomial algebra. The Lie group $GL(3) \times SU_L(2) \times SU_R(2)$ acts on linear space W spanned by generators

$p_1, \ldots, p_3, u^1, \ldots, u^3, z_1, z_2, w_1, w_2$. The representation W splits into a sum of irreducibles

$$W = < p_1, \ldots, p_3 > \oplus < u^1, \ldots, u^3 > \oplus < z_1, z_2 > \oplus < w_1, w_2 >$$

The space $< p_1, \ldots, p_3 >$ transforms according fundamental representation V of $GL(3)$; $< u^1, \ldots, u^3 >$ transforms by contragradient representation V^* of $GL(3)$; in both cases $SU_L(2) \times SU_R(2)$ factor acts trivially. The space $< z_1, z_2 >$ is irreducible two-dimensional representation T_L of $SU_L(2)$ on which $GL(3)$ acts via det; $< w_1, w_2 >$ is two-dimensional representation T_R of $SU_R(2)$ on which $GL(3)$ acts via det^{-1}. An algebra $\tilde{A}$ is defined as

$$\tilde{A} = \mathbb{C}[p_1, \ldots, p_3, u^1, \ldots, u^3, z_1, z_2, w_1, w_2]/I \tag{20}$$

where the ideal I is generated by a single equation

$$p_1 u^1 + p_2 u^2 + p_3 u^3$$

An algebra A is a subalgebra of $\tilde{A}$ generated by monomials of the form $p_\alpha z_i \quad (1 \leq \alpha \leq 3, 1 \leq i \leq 2)$, $u^\beta w_j \quad (1 \leq \beta \leq 3, 1 \leq j \leq 2)$ One can interpret algebra A as a multigraded Serre algebra corresponding to a collection of homogeneous line bundles over homogeneous space $F(1,2) \times \mathbb{C}P^1 \times \mathbb{C}P^1$. We use notations $F(1,2)$ for the space of flags in in three dimensional complex vector space. The symbol $\mathbb{C}P^1$ stands for one- dimensional complex projective space.

Let $\Lambda[\pi_{\alpha i}]$ $(1 \leq \alpha \leq 3, 1 \leq i \leq 2)$ be a Grassmann algebra on 6 anticommuting variables spanning $V \otimes T_L$ representation of $GL(3) \otimes SU_L(2)$. A Grassmann algebra $\Lambda[\psi_j^\beta]$ $(1 \leq \beta \leq 3, 1 \leq j \leq 2)$ is generated by $V^* \otimes T_R$. Introduce an algebra $\tilde{R}_{N=3}$

$$\tilde{A} \otimes \Lambda[\pi_{\alpha i}] \otimes \Lambda[\psi_j^\beta]$$

The algebra $\tilde{R}_{N=3}$ carries a differential d, defined by the formula

$$d(\pi_{\alpha i}) = p_\alpha z_i \quad (1 \leq \alpha \leq 3, 1 \leq i \leq 2) \tag{21}$$

$$d(\psi_j^\beta) = u^\beta w_j \quad (1 \leq \beta \leq 3, 1 \leq j \leq 2) \tag{22}$$

$$d(p_\alpha) = 0 \tag{23}$$

$$d(u^\beta) = 0 \tag{24}$$

$$d(z_i) = 0 \tag{25}$$

$$d(w_j) = 0 \tag{26}$$

It is clear that the algebra

$$R_{N=3} = A \otimes \Lambda[\pi_{\alpha i}] \otimes \Lambda[\psi_j^\beta] \tag{27}$$

is a differential graded subalgebra of $\tilde{R}_{N=3}$.

The claim is that the algebra $R_{N=3}$ is quasiisomorphic to Ω_0 and B_0. It is proved in the Appendix B.

5 $\mathbf{A}_\infty$ -algebras and gauge theories.

It is easy to formulate gauge theories in terms of $A_\infty-$ algebras.

Let us consider for example 10D SUSY YM theory reduced to a point. Corresponding action functional has the following form

$$S_{IKKT}(A,\chi) = tr(-\frac{1}{4}\delta^{ij}\delta^{kl}[A_i, A_k][A_j, A_l] + \frac{1}{2}\Gamma^i_{\alpha\beta}[A_i, \chi^\alpha]\chi^\beta)$$

Here A_i $i = 1\dots 10$ is a set of even $N \times N$ matrices, χ^α , $\alpha = 1\dots 16$ is a set of odd $N \times N$ matrices

The functional S_{IKKT} is invariant with respect to infinitesimal gauge transformations

$$\delta A_i = [A_i, \varepsilon]; \delta\chi^\alpha = [\chi^\alpha, \varepsilon]$$

We can extend it to the solution of BV master equation in standard way

$$S = S_{IKKT} + trA^{*i}[A_i, c] + tr\chi^*_\alpha[\chi^\alpha, c] + \frac{1}{2}tr[c, c]c^*$$

Corresponding nilpotent vector field Q can be written in the following way

$$QA^{*l} = \delta^{ij}\delta^{kl}[A_i, [A_j, A_k]] - \frac{1}{2}\Gamma^l_{\alpha\beta}\{\chi^\alpha, \chi^\beta\} - [A^{*l}, c] \tag{28}$$

$$Q\chi^*_\alpha = -\Gamma^i_{\alpha\beta}[A_i, \chi^\beta] - [\chi^*_\alpha, c] \tag{29}$$

$$Qc^* = -[A^{*i}, A_i] - [\chi^*_\alpha, \chi^\alpha] + [c, c^*] \tag{30}$$

$$Qc = \frac{1}{2}[c, c] \tag{31}$$

$$QA_i = [A_i, c] \tag{32}$$

$$Q\chi^\alpha = [\chi^\alpha, c] \tag{33}$$

Equations of motion in BV-formalism are obtained as conditions of vanishing of RHS of (28- 33). Using the connection between Q-manifolds and L_∞-algebras described in Appendix C and considering Taylor series at a point $A_i = \chi^\alpha = A^{*l} = \chi^*_\alpha = c^* = c = 0$ we identify equations of motion with Maurer-Cartan equations for some $L_\infty-$ algebra. In our case this L_∞-algebra can be obtained from $A_\infty-$ algebra $\mathcal{A}_{IKKT}$ by taking tensor product with matrix algebra and passing to corresponding $L_\infty-$ algebra. (This is true for equations of motion in any gauge theory.) Regarding $A_k, \chi^\alpha, c, A^{*k}, \chi^*_\alpha, c^*$ as formal noncommutative variables we can interpret (28, 29, 30, 31, 32, 33) as a definition of the algebra $\mathcal{A}_{IKKT}$. More precisely the algebra $\mathcal{A}_{IKKT}$ can be considered as vector space spanned by $A_k, \chi^\alpha, c, A^{*k}, \chi^*_\alpha, c^*$ with operations m_2

(multiplication), m_3(Massey product) defined by the following formulas:

$$m_2(\chi^\alpha, \chi^\beta) = \Gamma_k^{\alpha\beta} A^{*k} \tag{34}$$

$$m_2(\chi^\alpha, A_k) = m_2(A_k, \chi^\alpha) = \Gamma_k^{\alpha\beta} \chi^*_\beta \tag{35}$$

$$m_2(\chi^\alpha, \chi^*_\beta) = m_2(\chi^*_\beta, \chi^\alpha) = c^* \tag{36}$$

$$m_2(A_k, A^{*k}) = m_2(A^{*k}, A_k) = c^* \tag{37}$$

$$m_3(A_k, A_l, A_m) = \delta_{kl} A^{*m} - \delta_{km} A^{*l} \tag{38}$$

$$m_2(c, \bullet) = m_2(\bullet, c) = \bullet \tag{39}$$

$$\tag{40}$$

All other products are equal to zero. c is the unit of the $\mathrm{A}_\infty-$algebra (for all n -ary products ($n \geq 3$) if at least one entry equal to the unit the whole product vanishes).

All operations m_k with $k \neq 2, 3$ vanish.

One can prove that $\mathcal{A}_{IKKT}$ is quasiisomorphic to differential algebras Ω_0, B_0, etc described in (Sec. 4); taking into account that $m_1 = 0$ one can say that $\mathcal{A}_{IKKT}$ is a minimal model of these algebras. The proof can be based on diagram techniques for construction of minimal model developed in [24], [21] and reviewed in Appendix C. (One can use, for example, the embedding of homology of B_0 into B_0 constructed in [7].). Another proof will be given in [27].

Notice that the higher multiplications ($k > 2$) in $\mathrm{A}_\infty-$ algebra come from quartic and higher terms in the action functional. It is easy to represent the action functional S_{IKKT} in equivalent form containing at most cubic terms. (One should introduce the field strength $F_{ij} = [A_i, A_j]$ as an independent field.) This remark allows us to describe a smallest possible differential graded algebra quasiisomorphic to $\mathcal{A}_{IKKT}$.

Suppose V is a ten- dimensional vector representation of $Spin(10)$, S is a spinor representation, S^* is a dual spinor representation. The algebra $C^\bullet$ has the following graded components: $C_0 = \mathbb{C}$, $C_1 = C_2 = \{0\}$, $C_3 = V$, $C_4 = S$, $C_5 = \Lambda^2(V)$, $C_6 = \Lambda^2(V)$, $C_7 = S^*$, $C_8 = V$, $C_9 = C_{10} = \{0\}$, $C_{11} = \mathbb{C}$. All multiplication maps in this algebra are $Spin(10)$ equivariant as well as the

differential. The algebra is graded commutative.

The differential is equal to zero on all components except C_5, where $d : C_5 = \Lambda^2(V) \to \Lambda^2(V) = C_6$ is an isomorphism.

The space C_0 is generated by the unit. The multiplication defines a canonical pairing $C_i \otimes C_{11-i} \to C_{11}$, so we are dealing with a Frobenius algebra. Nontrivial multiplications are:

$$C_3 \otimes C_3 = V \otimes V \to \Lambda^2(V) = C_6$$

is the canonical projection,

$$C_3 \otimes C_4 = V \otimes S \xrightarrow{\Gamma} S^* = C_7$$

is specified by means of gamma matrices. The map.

$$C_5 \otimes C_4 = \Lambda^2(V) \otimes V \to V = C_8$$

is adjoint to inclusion $\Lambda^2(V) \to V \otimes V$.

One can extend the functional tr: $C_{11} \to \mathbb{C}$ to the map $C_{11} \otimes Mat_n \to \mathbb{C}$ using standard trace on algebra Mat_n; we keep the notation tr for extended functional.

Consider a tensor product $C^\bullet \otimes Mat_n$. The action functional corresponding to this algebra has the form

$$S(a) = tr(\frac{1}{2} a * d(a) + \frac{2}{3} a * a * a) \qquad (41)$$

for a field $a \in C^\bullet \otimes Mat_n$.

It is easy to check that this action functional is equivalent to the cubic form of S_{IKKT} .

6 SYM algebra and Koszul duality.

The algebra $F\left(\tilde{\mathcal{Q}}\right)$ of polynomials on the manifold of pure spinors can be considered as a graded algebra with generators $u^1, \ldots, u^n$ and quadratic relations

$$u^\alpha \Gamma^m_{\alpha\beta} u^\beta = 0, \quad u^\alpha u^\beta = u^\beta u^\alpha.$$

This fact permits us to study this algebra using well developed theory of quadratic algebras. Recall, that a graded algebra $A = \Sigma A_n$ is called a quadratic algebra if $A_0 = \mathbb{C}$, $W = A_1$ generates A and all relations follow from quadratic relations $\sum_{i,j} r_{ij}^k x^i x^j = 0$ where $x^1, \ldots, x^{\dim W}$ is a basis of $W = A_1$. The space of quadratic relations (the subspace of $W \otimes W$ spanned by $r_{ij}^1, r_{ij}^2, \ldots)$ will be denoted by R. We can say that $A_2 = W \otimes W/R$ and A is a quotient of free algebra (tensor algebra) ΣW^n with respect to the ideal generated by R. The dual quadratic algebra $A^!$ is defined as a free algebra $\Sigma \left(W^*\right)^{\otimes n}$ generated by W^* factorized with respect to the ideal generated by $R^\perp \subset W^* \otimes W^*$ (here $R^\perp$ stands for the subspace of $W^* \otimes W^* = (W \otimes W)^*$ that is orthogonal to $R \subset W \otimes W$). An important class of quadratic algebras consists of so called Koszul algebras. For definition of this notion and for more details about duality of quadratic algebras see [31], [22], [3] or [9].

The space of generators of $A = F\left(\tilde{\mathcal{Q}}\right)$ can be identified with the space of spinor representation S; the space R of relations is spanned by antisymmetric tensors and by ten-dimensional subrepresentation $V \subset (S \otimes S)^{sym}$. The space of generators of $A^!$ can be identified with S^* (dual spinors = right spinors); then $R^\perp$ is a subspace in $(S^* \otimes S^*)^{sym}$ orthogonal to V (it is generated by tensor squares of pure dual spinors). This means that $A^!$ is generated by $\lambda \in S^*$ with relations

$$\Gamma^{\alpha\beta}_{m_1,m_2,m_3,m_4,m_5} \left(\lambda_\alpha \lambda_\beta + \lambda_\beta \lambda_\alpha\right) = 0. \tag{42}$$

where $\Gamma^{\alpha\beta}_{m_1,\ldots,m_5}$ stands for the skew-symmetrized product of five $\Gamma-$matrices

$$\Gamma^{\alpha\delta_1}_{m_1} \Gamma_{m_2 \delta_1 \delta_2} \Gamma^{\delta_2 \delta_3}_{m_3} \Gamma_{m_4 \delta_3 \delta_4} \Gamma^{\delta_4 \beta}_{m_5} \tag{43}$$

Notice, that these relations can be considered as conditions on (super) commutator $[\lambda_\alpha, \lambda_\beta]_+ = \lambda_\alpha \lambda_\beta + \lambda_\beta \lambda_\alpha$. Hence we can regard (42) as defining relations of a (super) Lie algebra L and the algebra $A^! = F\left(\tilde{\mathcal{Q}}\right)^!$ can be regarded as its enveloping algebra: $F\left(\tilde{\mathcal{Q}}\right)^! = U(L)$.

The construction of the algebra $A = F\left(\tilde{\mathcal{Q}}\right)$ is a particular case of the

following general construction. Let us consider a holomorphic line bundle α over complex manifold M and the space Γ_n of holomorphic sections of the tensor power $\alpha^{\otimes n}$ of the bundle α. The natural multiplication $\Gamma_m \otimes \Gamma_n \to \Gamma_{m+n}$ defines a structure of commutative associative graded algebra on the direct sum $\Gamma = \bigoplus_{n \geq 0} \Gamma_n$; this algebra is called Serre algebra corresponding to the pair (M, α). It is easy to generalize this construction to the case when we have a collection of line bundles $\alpha_1, \ldots, \alpha_k$. In this case the direct sum of spaces $\Gamma_{n_1,\ldots,n_k}$ of sections of line bundles $\alpha_1^{\otimes n_1} \otimes \cdots \otimes \alpha_k^{\otimes n_k}$ $(n_1, \ldots, n_k \geq 0)$ carries a structure of commutative algebra which is called a multigraded Serre algebra. When $M = G/P$ is a compact complex homogeneous space of semisimple Lie group G and α is a homogeneous line bundle one can prove that corresponding algebra is quadratic and that this algebra is Koszul algebra [9]. The algebra $A = F\left(\tilde{\mathcal{Q}}\right)$ can be obtained as a Serre algebra corresponding to $SO(10)/U(5)$, hence it is Koszul algebra.

Now we can consider the dual algebra to the algebra B. The algebra B is also a quadratic algebra with generators u^α, θ^α and relation

$$u^\alpha \Gamma^m_{\alpha\beta} u^\beta = 0,\ u^\alpha u^\beta = u^\beta u^\alpha,\ \theta^\alpha\theta^\beta + \theta^\beta\theta^\alpha = 0, u^\alpha\theta^\beta = \theta^\beta u^\alpha$$

Its dual algebra $B^!$ is an algebra with generators λ_α, t_α and relations

$$\Gamma^{\alpha\beta}_{m_1,\ldots m_5}\left(\lambda_\alpha\lambda_\beta + \lambda_\beta\lambda_\alpha\right) = 0,\ t_\alpha t_\beta - t_\beta t_\alpha = 0,\ \lambda_\alpha t_\beta - \lambda_\beta t_\alpha = 0.$$

Again these relations involve only (anti)commutators and can be considered as defining relation of (super) Lie algebra $\mathbb{L}$. The algebra $B^!$ can be regarded as an enveloping algebra of $\mathbb{L}$. The differential acting on B is defined by formulas $d\theta^\alpha = u^\alpha,\ du^\alpha = 0$. The dual differential acting on $B^!$ obeys $d\lambda_\alpha = t_\alpha,\ dt_\alpha = 0$. It is clear from this description that

$$A_m = \Gamma^{\alpha\beta}_m \lambda_\alpha\lambda_\beta \ \text{ and } \ \chi^\alpha = \Gamma^{\alpha\beta}_m \left[\lambda_\beta, A^m\right]$$

satisfy $dA_m = 0,\ d\chi^\alpha = 0$.

Let us denote by SYM the subalgebra of the algebra $B^!$ generated by A_m and χ^α.

It is easy to check that generators of SYM obey relations

$$[A_i,[A_i,A_k]]-\frac{1}{2}\Gamma^k_{\alpha\beta}\{\chi^\alpha,\chi^\beta\}=0 \tag{44}$$

$$-\Gamma^i_{\alpha\beta}[A_i,\chi^\beta]=0 \tag{45}$$

that follow from (42). Comparing (44, 45) with equations of motion of 10D SUSY YM reduced to a point we obtain that the solutions to these equations of motion in $n\times n$ matrices can be identified with $n-$dimensional representations of the algebra SYM.

It is easy to check that $SYM\subset Kerd\subset B_0^!$. This inclusion determines a homomorphism of SYM into homology $H\left(B_0^!\right)$ of $B_0^!$; one can prove that this homomorphism is an isomorphism (see [27]). This permits us to identify SYM with $H\left(B_0^!\right)$ and to say that the embedding of SYM into $B_0^!$ is a quasiisomorphism.

Notice, that an analog of algebra SYM for non-supersymmetric Yang-Mills theory (the Yang-Mills algebra YM) was considered in [30] and [11]; the paper [11] contains, in particular, the construction of Koszul dual to YM.

7 Appendix A. Quasiisomorphism of Dolbeault and Berkovits algebras.

Let us sketch the proof of quasiisomorphism of algebras $(\Omega_0,\overline{\partial})$ and (B_0,d); the proof of quasiisomorphism between $(\Omega,\overline{\partial})$ and (B,d) is similar (see [25] for more details).

Let us start with the analysis of the algebra $F(\widetilde{\mathcal{Q}})$ of polynomial functions on the manifold of pure spinors $\widetilde{\mathcal{Q}}$. Elements of $F(\widetilde{\mathcal{Q}})$ can be characterized as holomorphic functions on $\widetilde{\mathcal{Q}}$ that have polynomial growth at infinity and are bounded on bounded subsets of $\widetilde{\mathcal{Q}}$. We can work with the manifold $\widehat{\mathcal{Q}}$ instead of $\widetilde{\mathcal{Q}}$. (Recall that $\widetilde{\mathcal{Q}}$ is a total space of principal $\mathbb{C}^*$-bundle α over $\mathcal{Q}$ and $\widehat{\mathcal{Q}}$

is a total space of line bundle $\widehat{\alpha}$ that is associate to this $\mathbb{C}^*$-bundle.) Then we can characterize $F(\widetilde{\mathcal{Q}})$ as the space of holomorphic functions on $\widehat{\mathcal{Q}}$ having polynomial growth at infinity. (Every continuous function of $\widehat{\mathcal{Q}}$ determines a function on $\widetilde{\mathcal{Q}}$ that is bounded on every bounded domain. This follows from corresponding fact about $\mathbb{C}$ and $\mathbb{C}^*$.)

Let us denote by $(\Omega(\widehat{\mathcal{Q}}), \overline{\partial})$ the algebra of $(0,k)$-forms on $\widehat{\mathcal{Q}}$ having at most polynomial growth at infinity. This algebra is quasiisomorphic to $F(\widetilde{\mathcal{Q}})$; more precisely the natural embedding of $F(\widetilde{\mathcal{Q}})$ into $\Omega(\widehat{Q})$ is a $Spin(10,\mathbb{C})$-equivariant quasiisomorphism. We identified already $H^0(\Omega(\widehat{\mathcal{Q}}), \overline{\partial})$ with $F(\widetilde{\mathcal{Q}})$; it remains to verify that $H^k(\Omega(\widehat{\mathcal{Q}}), \overline{\partial}) = H^{0,k}(\widehat{\mathcal{Q}}) = 0$ for $k > 0$. The calculation of $H^{0,k}(\widehat{\mathcal{Q}})$ can be reduced to the calculation of Dolbeault cohomology $H^{0,k}(\widehat{\beta}^{\otimes n})$ $n \geq 0$ of line bundles $\widehat{\beta}^{\otimes n}$over $\mathcal{Q}$. The line bundle $\widehat{\beta}$ is dual to $\widehat{\alpha}$; these cohomology vanish unless $k = 0$.

Using the quasiisomorphism between $(\Omega(\widehat{\mathcal{Q}}), \overline{\partial})$ and $F(\widetilde{\mathcal{Q}})$ we obtain that the natural embedding of $B_0 = (F(\widetilde{\mathcal{Q}}) \otimes \Lambda(S^*), d)$ into the algebra $(\ \Omega(\widehat{\mathcal{Q}}) \otimes \Lambda(S^*), \overline{\partial} + d)$ is a quasiisomorphism. (To check that the latter algebra has the same cohomology as B_0 one can consider it as a bicomplex with differentials $\overline{\partial}$ and d and prove that the corresponding spectral sequence degenerates, hence the cohomology is equal $H(H(\overline{\partial}), \widetilde{d})$ where $H(\overline{\partial})$ stands for the cohomology of $\Omega(\widehat{\mathcal{Q}}) \otimes \Lambda(S^*)$ with respect to $\overline{\partial}$ and $\widetilde{d}$ denotes the differential on $H(\overline{\partial})$ induced by d.)

The algebra $(\ \Omega(\widehat{\mathcal{Q}}) \otimes \Lambda(S^*), \overline{\partial} + d)$ is quasiisomorphic to the algebra $\Omega(\widehat{\mathcal{Q}} \times \Pi S)$ of $(0,k)$-forms on the manifold $\widehat{\mathcal{Q}} \times \Pi S$ with differential $\overline{\partial} + d$. (This follows from decomposition $\Omega(\widehat{\mathcal{Q}} \times \Pi S) = \Omega(\widehat{\mathcal{Q}}) \otimes \Lambda(S^*) \otimes \Lambda(\overline{S}^*) \otimes S(\overline{S}^*)$. The natural projection $\widehat{\mathcal{Q}} \times \Pi S$ onto $\mathcal{R}_0$ induces a homomorphism of $(\Omega_0, \overline{\partial})$ into $(\ \Omega(\widehat{\mathcal{Q}} \times \Pi S), \overline{\partial} + d)$; one can check that this homomorphism is a quasiisomorphism. To construct a map $\widehat{\mathcal{Q}} \times \Pi S \to \mathcal{R}_0$ we interpret $\widehat{\mathcal{Q}} \times \Pi S$ as a total space of a bundle over $\mathcal{Q} = SO(10,\mathbb{R})/U(5)$ that corresponds to representation λ of $U(5)$ on $\mathbb{C} \times \Pi S$; it is easy to check that λ can be expressed in terms of vector

representation W of $U(5)$ as

$$\lambda = (\det W)^{-1/2}, (\mathbb{C} + \Lambda^2(W) + \Lambda^4(W)) \otimes (\det W)^{-1/2}) \tag{46}$$

Denote a generator of linear space $(\det W)^{-1/2}, \{0\}) \subset \lambda$ by ξ , the generator of $(\{0\}, detW^{-1/2} + \{0\}) \subset \lambda$ by x. As we have seen $\mathcal{R}_0$ can be interpreted in similar way by means of representation

$$\varkappa = (\Lambda^2(W) + \Lambda^4(W)) \otimes (\det W)^{-1/2}$$

The obvious intertwiner between λ and $\varkappa$ gives rise to the map we need. This map induces a homomorphism of $\Omega_0 = \Omega(\mathcal{R}_0)$ into $\Omega(\widehat{\mathcal{Q}} \times \Pi S)$. To prove that this homomorphism is a quasiisomorphism we notice that in coordinates (46) the differential d is given by the formula $x\frac{\partial}{\partial\xi} + \bar{x}\frac{\partial}{\partial\bar{\xi}}$. It means that locally x and ξ ($\bar{x}$ and $\bar{\xi}$)are contractible pairs and locally over $\mathcal{Q}$ the inclusion i is quasiisomorphism. One can globalize this statement using a partition of unity.

8 Appendix B.
Quasiisomorphic Koszul complexes.

Let us consider a differential (super)commutative associative algebra $\mathcal{K}$ and n even elements $f_1, ..., f_n \in \mathcal{K}$ obeying $df_1 = ... = df_n = 0$. Then we can define differential algebra $\mathcal{K}^{f_1,...,f_n}$ (Koszul complex) adding "ghosts" $c_1, ..., c_n$ and -extending the differential by the formula $dc_i = f_i$. (More formally, $\mathcal{K}^{f_1,...,f_n}$ is defined as a tensor product of $\mathcal{K}$ with exterior algebra generated by $c_1, ..., c_n$.) Notice that the algebra $\mathcal{K}^{f_1,...,f_n}$ does not change if we replace $f_1, ..., f_n$ with their linear combinations spanning the same ideal. One says that $f_1, ..., f_n$ is a regular sequence if for every $r = 0, ..., n-1$ the elements $f_{r+1}, ..., f_n$ do not divide zero in $\mathcal{K}/(f_1, ..., f_r)$. Here $(f_1, ..., f_r)$ stands for the ideal generated by $f_1, ..., f_r$. If the sequence $f_1, ..., f_n$ is regular then the natural homomorphism $\mathcal{K}^{f_1,...,f_n}$ into $\mathcal{K}/(f_1, ..., f_r)$ is a quasiisomorphism. (This statement is well known for the case when the differential in $\mathcal{K}$ is trivial. Then $\mathcal{K}^{f_1,...,f_n}$ is called Koszul resolution

of $\mathcal{K}/(f_1, ..., f_r)$. The generalization to the case of non-trivial differential in $\mathcal{K}$ is obvious.)

Let us assume now that the sequence $f_1, ..., f_n$ is not regular, but its subsequence $f_{r+1}, ..., f_n$ is regular. Then one can prove that the algebra $\mathcal{K}^{f_1,...,f_n}$ is quasiisomorphic to the algebra $\mathcal{K}^{f_1,...,f_r}/(f_{r+1}, ..., f_n)$. This follows immediately from the remark that

$$\mathcal{K}^{f_1,...,f_n} = (\mathcal{K}^{f_1,...,f_r})^{f_{r+1},...,f_n}.$$

We can apply the above statement to construct algebras that are quasiisomorphic to Berkovits algebra. We start with the algebra $\mathcal{K} = F(\widetilde{\mathcal{Q}})$ of polynomial functions on the space of pure spinors: $\mathcal{K} = \mathbb{C}[u^1, ..., u^{16}]/(u\Gamma u)$. It is easy to check that Berkovits algebra (B_0, d) is isomorphic to $\mathcal{K}^{u_1,...,u_{16}}$. We can verify that B_0 is quasiisomorphic to the algebra $R_{N=3}$ using the following construction: we notice that the Lie algebra $\mathfrak{g}l(3) \times \mathfrak{s}u(2)_L \times \mathfrak{s}u(2)_R$ is a subalgebra of $\mathfrak{s}o(10)$. The spin representation spanned by $< u^1, ..., u^{16} >$ restricted on $\mathfrak{g}l(3) \times \mathfrak{s}u(2)_L \times \mathfrak{s}u(2)_R$ splits into $V \otimes T_L \oplus V^* \otimes T_R \oplus T_L \oplus T_R$ where V is a fundamental representation of $\mathfrak{g}l(3)$, V^* is its dual, T_L is a two-dimensional representation of $\mathfrak{s}u(2)_R$. The ten-dimensional representation of $Spin(10)$ decomposes as $V + V^* + T_L \otimes T_R$. It is useful to write down pure spinor equation in this context. The coordinates of spinor are $t_{\alpha i}, s^\beta_j, z_i, w_j$, where $1 \le ij \le 2$, $1 \le \alpha, \beta \le 3$. The equations are

$$t_{\alpha 1} z_2 - t_{\alpha 2} z_1 = det(s^\beta_j) \tag{47}$$

$$s^\alpha_1 w_2 - s^\alpha_2 w_1 = det(t_{\alpha i}) \tag{48}$$

$$s^\alpha_i t_{\alpha j} = 0 \tag{49}$$

Expressions $det(s^\beta_j)$, $det(t_{\alpha i})$ stand for vector formed by principal minors of 2×3 matrices.

It can be shown using Maple package Gröbner that the algebra $\mathcal{K}$ is a free $\mathbb{C}[w_i, z_j]$ module, i.e. it is equal to $\mathcal{N} \otimes \mathbb{C}[w_i, z_j]$ for some linear space $\mathcal{N}$. It is well known(e.g. see [23]) that in the case of polynomial algebra $\mathcal{C} = \mathbb{C}[x_1, \ldots, x_n]$ the cohomology $H^\bullet(\mathcal{C}^{x_1,...,x_r})$ is equal to $\{0\}$ in all degrees but

zero where it is equal to $\mathcal{C}/(x_1, \dots, x_r)$. This implies that $H^\bullet(V \otimes A^{x_1,\dots,x_r}) = V \otimes H^\bullet(A^{x_1,\dots,x_r})$. For the algebra at hand this means that $H^\bullet(\mathcal{K}^{w_i,z_j}) = H^\bullet(V \otimes \mathbb{C}[w_i, z_j]^{w_i,z_j}) = V \otimes H^\bullet(\mathbb{C}[w_i, z_j]^{w_i,z_j})$. Hence the complex $\mathcal{K}^{w_i,z_j}$ has cohomology only in zero degree.

Define a map $\rho : \mathcal{K}/_{(w_i,z_j)} \to \tilde{A}$ by the rule

$$\rho(t_{\alpha i}) = p_\alpha z_i$$
$$\rho(s_j^\beta) = u^\beta w_j$$

(recall that $\tilde{A}$ was defined in (Sec. 4) by means of (Equ. 20)) It is easy to check correctness of this map. Its image is a subalgebra $A \subset \tilde{A}$. We interpreted the algebra A as a Serre algebra. Using a theorem that a multigraded Serre algebra of a compact homogeneous space of a semisimple group is quadratic (see [9] and references therein) , it is not hard to show that the map ρ is an isomorphism of $\mathcal{K}/_{(w_i,z_j)}$ and algebra A. This means that differential graded algebra $\mathcal{K}^{w_i,z_j}$ is quasiisomorphic to A , hence $\mathcal{K}^{u_1,\dots,u_{16}}$ is quasiisomorphic to $R_{N=3}$. D.

Piontkovski pointed out to us that the length of a maximal regular sequence for algebras with duality in cohomology of the Koszul complex (so called Gorenstein algebras) is equal to the dimension of the underlying affine manifold. Every regular sequence can be extended to the maximal one. Maximal regular sequences form a dense set (in a suitable topology) among all sequences of the same length. For the algebra $F(\widetilde{\mathcal{Q}})$ the maximal length of regular sequence is equal to 11. The maximal regular sequence leads to differential algebra closely related to the algebra $C^\bullet$ described in (Sec. 5).

9 Appendix C.
$\mathbf{L_\infty}$ and $\mathbf{A_\infty}$ algebras

The BV formalism is based on consideration of classical master equation $\{S, S\} = 0$ on odd symplectic manifold (P-manifold). Every solution of classical master equation generates an odd vector field Q corresponding to the first order differ-

ential operator $Q : \varphi \to \{\varphi, S\}$. This remark permits us to say that a geometric counterpart of solution of classical master equation is a PQ-manifold, i.e. a P-manifold X equipped with an odd vector field Q obeying $Q^2 = 0$ and $L_Q\sigma = 0$. Here σ is an odd symplectic form on X and L_Q is a Lie derivative along vector field Q. A manifold equipped with an odd vector field Q obeying $Q^2 = 0$ is called a Q-manifold. One can say that PQ-manifold is equipped with structures of P-manifold and Q-manifold and these structures are compatible.

Let R denote the zero locus of Q on Q-manifold X. Then the differential Q_x of Q at a point $x \in R$ can be considered as a linear operator acting on a tangent space T_xQ. One can identify $KerQ_x$ with the tangent space T_xR, the spaces ImQ_x specify a foliation of R; we denote a space of leaves of this foliation by $\tilde{R}$. If Q corresponds to the solution S of the classical master equation, the zeros of Q coincide with extrema of S (solutions to the classical equations of motion) and $\tilde{R}$ can be interpreted as the moduli space of solutions to the classical equations of motion.

There is some freedom in presenting classical mechanical system in BV-form. This freedom leads to notion of equivalent (quasiisomorphic) Q-manifolds and equivalent (quasiisomorphic)$PQ-$ manifolds. Let us suppose that f is a map of Q- manifold into Q- manifold X' that is compatible with vector field Q i.e. $f_*Q = Q$. (We use the same notation for nilpotent vector field on X and on X'.) Then the zero locus R of Q on X goes to zero locus R' of Q on X' and $\tilde{R}$ is mapped into $\tilde{R}'$. We say that f is quasiisomorphism between X and X' if correspondence between $\tilde{R}$ and $\tilde{R}'$ is bijective.

The notion of Q-manifold is closely related to the notion of L_∞ -algebra. Namely if p is a point of M where the vector field Q vanishes the coefficients of Taylor expansion of Q at the point m specify a structure of L_∞-algebra on the tangent space $E = \Pi T_{x_0}(M)$. More precisely, we consider the (formal) Taylor series

$$Q^a(z) = \sum_{k=1}^{\infty} \sum_{i_1,\dots,i_k} {}^{(k)}m^a_{i_1,\dots,i_k} z^{i_1} \dots z^{i_k} \tag{50}$$

of the vector field Q with respect to the local coordinate system $z^1, ..., z^N$ centered in $p \in R$. Coefficients in (50) are symmetric in the sense of superalgebra, i.e. symmetric with respect to transpositions of two even indexes or an even index with an odd one and antisymmetric with respect to transpositions of two odd indexes ("parity of index i" means parity of the corresponding coordinate z^i). We assume that the field Q is smooth.

It is easy to check that the condition $\{Q, Q\} = 0$ is equivalent to the following relations for the coefficients ${}^{(k)}m^a_{i_1,\ldots,i_k}$:

$$\sum_{\substack{k,l=0 \\ k+l=n+1}}^{n+1} \sum_{p=1}^{k} \sum_{\substack{perm \\ i_1,\ldots,\hat{i}_p,\ldots,i_k \\ j_1,\ldots,j_l}} \pm {}^{(k)}m^a_{i_1,\ldots,i_p,\ldots,i_k} {}^{(l)}m^{i_p}_{j_1,\ldots,j_l} = 0 \tag{51}$$

where $\pm$ depends on the particular permutation.

Let us write the first relations, assuming for simplicity that ${}^{(3)}m = 0$. It is more convenient to use instead of ${}^{(k)}m$ defined above the following objects:

$$d^a_b = {}^{(1)}m^{a'}_{b'}$$

and

$$f^a_{bc} = \pm {}^{(2)}m^{a'}_{b'c'}$$

where

$$\pm = (-1)^{(\epsilon(a')+1)\epsilon(b')}$$

Here $\epsilon(a)$ denotes the parity of the index a, and the parity of the indexes a, b and c is opposite to the parity of the corresponding indexes in the right hand side: $\epsilon(a) = \epsilon(a') + 1$, etc. From these formulas one can easily get the symmetry condition for f^a_{bc}:

$$f^a_{bc} = (-1)^{\epsilon(b)\epsilon(c)+1} f^a_{cb} \tag{52}$$

Then we have

$$d^m_b d^c_m = 0 \tag{53}$$

$$f^r_{mb} d^m_c + (-1)^{\epsilon(b)\epsilon(c)} f^r_{mc} d^m_b + d^r_m f^m_{cb} = 0 \tag{54}$$

$$f^r_{mb}f^m_{cd} + (-1)^{(\epsilon(b)+\epsilon(d))\epsilon(c)} f^r_{mc}f^m_{db} + (-1)^{(\epsilon(c)+\epsilon(d))\epsilon(b)} f^r_{md}f^m_{bc} = 0 \tag{55}$$

Relation (55) together with relation (52) means that the f^a_{bc} can be considered as structure constants of a super Lie algebra. The matrix d^a_b determines an odd linear operator d satisfying $d^2 = 0$ (this follows from (53)). The relation (54) can be considered as a compatibility condition of the Lie algebra structure and the differential d.

In a more invariant way we can say that the coefficients ${}^{(k)}m$ determine an odd linear map of k^{th} symmetric tensor power of $S^k(T_pM)$ into T_pM. This map induces a map $m_k : S^k(V) \to V$, where $V = \Pi T_pM$. Here m_k is odd for odd k and even for even k. The map m_1 determines a differential in V and m_2 determines a binary operation there. The relations (52-55) show that in the case when ${}^{(3)}m = 0$ the space V has a structure of a differential Lie (super)algebra.

If ${}^{(3)}m \neq 0$ the Jacobi identity (55) should be replaced with the identity involving ${}^{(3)}m$ (the so called homotopy Jacobi identity). However taking homology $H(V)$ with respect to the differential $m_1 = d$ we get a Lie algebra structure on $H(V)$.

One can consider m_k as a k-ary operation on V. Relation (51) can be rewritten as a set of relations on the operations m_k. A linear space provided with operations m_k satisfying these relations is called a L_∞-algebra or strongly homotopy Lie algebra. (The name homotopy Lie algebra is used when there are only m_1, m_2 and m_3 satisfying the corresponding relations.) The notion of strong homotopy algebra was introduced by J. Stasheff who realized also that this algebraic structure appears in string field theory [36].

The construction above gives a structure of L_∞-algebra to the space $V = \Pi T_pM$ where p is a stationary point of an odd vector field Q satisfying $\{Q, Q\} = 0$. It is possible also to include an "operation" ${}^{(0)}m$ in the definition of L_∞-algebra; then the structure of an L_∞-algebra arises in the space $V = \Pi T_pM$ at every point p of the Q-manifold M. However we will not use this modified definition.

One can say that L_∞-algebra is a formal Q-manifold. (The space of functions on formal $(m|n)$-dimensional supermanifold can be identified with supercommutative algebra $\hat{C}^{m|n}$ of formal series with respect to commuting variables $x^1, \ldots, x^m$ and anticommuting variables $\zeta^1, \ldots, \zeta^n$. The algebra $\hat{C}^{m|n}$ can be considered as completion of free nonunital supercommutative algebra $C^{m|n}$ generated by even elements $x^1, \ldots, x^m$ and odd elements $\zeta^1, \ldots, \zeta^n$. See [20] for discussion of formal and partially formal supermanifolds.)

Maurer-Cartan equation in L_∞-algebra has the form

$$\sum_{k=1}^{\infty} \frac{1}{k!} m_k(a, \ldots, a) = 0$$

Interpreting L_∞-algebra as a Q-manifold one can identify the set of solutions to the Maurer-Cartan equation with zero locus R of vector field Q. The moduli space of solutions to the Maurer-Cartan equation is obtained from this set by means of some identification; it corresponds to $\tilde{R}$.

To talk about solutions to Maurer-Cartan equation one should have a notion of a point of formal manifold; see [20] for discussion of this notion.

Notice that in the case when the vector field Q is polynomial (i.e. we have only finite number of operations m_k) we can work with a free algebra T_n instead of its completion $\hat{T}_n$.

In the case of graded L_∞-algebra we can also work with the free algebra T_n.

If X is a PQ-manifold and the field Q vanishes at x_0 we can say that L_∞-algebra constructed above is equipped with an odd nondegenerate inner product. More precisely there exists such an odd bilinear form $< .,. >$ on E that $< a_0, m_k(a_1, ..., a_k) >= \mu_k(a_0, a_1, ..., a_k)$ is cyclically (graded)symmetric (to construct such an inner product we notice that in a neighborhood of x_0 we can find coordinates in a such a way that coefficients of symplectic form are constant). We can identify an L_∞-algebra algebra with a nondegenerate inner product with a formal PQ-manifold. Using this identification we can say that Maurer-Cartan equations are equations of motion corresponding to the action

functional

$$S(a) = \sum_{k=1}^{\infty} \frac{1}{k!}\mu_k(a, \ldots, a) \tag{56}$$

This statement is a particular case of general fact that the equation $Qx = 0$ can be regarded as an equation of motion corresponding to BV-action functional S.

The action (56) can be considered as generalization of Chern-Simons action. (If an L_∞algebra is a differential Lie algebra, i.e. $\mu_k = 0$ for $n \geq 3$, then (56) looks as standard Chern-Simons action. One can prove that any L_∞algebra is quasiisomorphic to differential Lie algebra. This means that every action can be represented in Chern-Simons form.)

The notion of Q-manifold can be defined also for noncommutative spaces. An associative $\mathbb{Z}_2$ graded algebra can be regarded as an algebra of functions on noncommutative (super)space. Vector fields are identified with derivations (infinitesimal automorphisms), odd vector fields with odd (parity changing) derivations. Noncommutative Q-manifold is a differential associative $\mathbb{Z}_2$ graded algebra, i.e an algebra equipped with an odd (parity reversing) derivation Q obeying $Q^2 = 0$.

One can define an A_∞-algebra as a formal non-commutative Q -manifold. A formal non-commutative manifold for us will mean a manifold with coordinates $x^1, ..., x^n$ that do not satisfy any relations. In other words the algebra of functions on such manifold is a completion $\widehat{T}_n$ of free algebra T_n generated by $x^1, ..., x^n$. More precisely, $\widehat{T}_n$ consists of all infinite series in noncommuting variables $x^1, ..., x^n$. We assume that coordinates $x^1, ..., x^n$ and hence the algebra they generate A are $\mathbb{Z}_2$- graded (i.e. some of the coordinates are considered as even and some as odd). Notice that we did not assume that the algebra T_n is unital; in other word a series belonging to $\widehat{T}_n$ cannot contain a constant term. A vector field is identified with a derivation (infinitesimal automorphism) of this algebra. A vector field Q specifying an A_∞-algebra structure is an odd derivation obeying $Q^2 = 0$. It is sufficient to specify Q on generators $x^1, ..., x^n$:

$$Qx^i = \sum m^i_{i_1,\dots,i_k} x^{i_1} \dots x^{i_n}$$

The vector field Q is uniquely determined by its "Taylor coefficients" $m^i_{i_1,\dots,i_k}$. We consider them as polylinear operations defined by the formula

$$m_k(e_{i_1}, ..., e_{i_k}) = \pm m^a_{i_1,\dots,i_k} e_a \tag{57}$$

We assume that elements $e_1, ..., e_n$ are in 1-1 correspondence with $x^1, ..., x^n$ but the parity of e_i is opposite to parity of x^i. If $m_k = 0$ for $k \geq 3$ then its follows from the above relations that m_1, m_2 specify of differential associative algebra. Conversely, every differential associative algebra can be considered as A_∞-algebra; the space $\hat{T}$ is dual to the space of Hochschild cochains and Q is induced by Hochschild differential.

The condition $Q^2 = 0$ can be rewritten as a sequence of quadratic equations

$$\sum_{i+j=n+1} \sum_{0 \leq l \leq i} \epsilon(l,j) m_i(a_0, ..., a_{l-1}, m_j(a_l, ..., a_{l+j-1}), a_{l+j}, ..., a_n) = 0$$

where $a_m \in A$, and $\epsilon(l,j) = (-1)^{j \sum_{0 \leq s \leq l-1} deg(a_s) + l(j-1) + j(i-1)}$. In particular, $m_1^2 = 0$.

In coordinate-free language we start with $\mathbb{Z}_2$-graded vector space V and define tensor algebra $\hat{T}(V)$ as completion of tensor algebra

$$T(V) = \bigoplus_{n \geq 0} V^{\otimes n}$$

We consider $\hat{T}(V)$ as topological $\mathbb{Z}_2$-graded algebra. A continuous derivation Q of $\hat{T}(V)$ is completely determined by its values on $V^{\otimes 1} \subset \hat{T}(V)$. In other words to specify Q one needs to fix linear maps $\mu_k : V \to V^{\otimes k}$. These maps determine $\mathbb{C}$-linear operations $m_k : A^{\otimes k} \to A$, where A stands for the space obtained from V^* by means of parity reversion: $A = \Pi V^*$. If $Q^2 = 0$, then the operations specify a structure of A_∞-algebra on A. Notice that since Q is odd, the parity of $m_k(a_1, \dots, a_k)$ is equal to $k(\sum dega_i)$ mod $\mathbb{Z}_2$, where $dega_i$ stands for the parity of a_i

An element 1 is called the unit of A_∞-algebra A if it is the unit of binary operation m_2 and in all higher order operations we have $m_k(\dots,1,\dots)=0$.

It is convenient to assume that A has no unit or that it is obtained from nonunital algebra by means of adjunction of a unit.

Considering L_∞-algebras as formal Q- manifolds we can define an L_∞-morphism of L_∞-algebras as a map of Q- manifolds which is compatible with Q . (One says that that a map $f: M \to M'$ of Q- manifolds is compatible with Q if $f_*Q = Q'$. We use here the same notation Q for vector fields specifying Q-structure on M and M'.) As in the case of definition of structure maps m_k in (57) of L_∞- algebra the components of L_∞-morphism can be read off from Taylor coefficients of the map f. This means that an L_∞-morphism from L to L' is a system of linear transformation $f_k : L^{\otimes k} \to L'$ obeying nonlinear equations that follow from compatibility between f and Q.

It is easy to check that $f_1 m_1 = m_1' f_1$. This means that L_∞-morphism induces a homomorphism of homology $H(L, m_1) \to H(L', m_1')$. One says that L_∞-morphism morphism is a quasiisomorphism if it induces an isomorphism on homology. All important notions are invariant with respect to quasiisomorphism; in particular, the moduli space of solutions to Maurer-Cartan equation does not change if we replace an algebra with quasiisomorphic algebra.

An L_∞-algebra L is called minimal if $m_1 = 0$. One can prove that every L_∞-algebra is quasiisomorphic to a minimal L_∞-algebra. (In other words, there exists a minimal L_∞- structure on $H(L, m_1)$ and a quasiisomorphism between L and $H(L, m_1)$ with this L_∞-structure.) More precisely, one can say that every L_∞-algebra is isomorphic to a direct sum of minimal algebra and trivial algebra. (We say that L_∞-algebra is trivial if it has only one non-zero operation m_1 and trivial homology.)

Two minimal L_∞-algebras are quasiisomorphic iff they are isomorphic (i.e. they are connected by invertible L_∞-morphism). On the other hand every L_∞-algebra is quasiisomorphic to differential Lie algebra (i.e. to L_∞-algebras algebra with $m_k = 0$ for $k \geq 3$). In other words quasiisomorphism (L_∞-quasiisomorphism) classes of differential Lie algebras can be identified with

quasiisomorphism classes of L_∞-algebras and isomorphism classes of minimal L_∞-algebras.

Similar statements are true for A_∞-algebras and differential associative algebras if we modify definitions in appropriate way.(For example, the definition of A_∞-morphism is analogous to the definition of L_∞-morphism.)

The above statements give us an explanation of the role of A_∞-algebras. Differential associative algebras are ubiquitous in physics. In particular, BRST operator can be considered as a differential on an appropriate algebra of operators. Usually two quasiisomorphic algebras are physically equivalent. This fact allows us to replace a differential algebra defined by BRST operator with much smaller A_∞-algebra on the space of observables (on BRST-cohomology). From the other side one can avoid the use of A_∞-algebras working with larger differential associative algebras.

There exists a diagram technique, that permits us to calculate explicitly the minimal model of A_∞-algebra or L_∞-algebra. More generally, let us denote by B an m_1-invariant $\mathbb{Z}_2$-graded subspace of a nonunital A_∞-algebra A. Let us assume that there exist a linear operator $P{:}A \to A$ obeying $P^2 = P$ that projects A onto B and commutes with m_1. We assume that the projection P is homotopic to the identity map, i.e. there exists an odd operator H obeying $1 - P = m_1 H + H m_1$. In these assumptions the embedding $i : B \to A$ and the projection $p : A \to B$ induce isomorphisms between homology of B and A. (The projection p is defined by the formula $P = i \circ p$.) Following [21] we define a structure of A_∞-algebra on B. If the differential m_1 acts trivially on B this construction gives a minimal model of A.

One can introduce a sequence of linear operations $m_n^B : B^{\otimes n} \to \Pi^n B$ in the following way

a) $m_1^B := d^B = p \circ m_1 \circ i$;

b) $m_2^B = p \circ m_2 \circ (i \otimes i)$;

c) $m_n^B = \sum_T \pm m_{n,T}, n \geq 3$.

Here the summation is taken over all oriented planar trees T with $n+1$ tails vertices (including the root vertex), such that the number of ingoing edges of

every internal vertex of T is at least 2. The linear map $m_{n,T} : B^{\otimes n} \to \Pi^n B$ where Π stands for parity reversion can be described in the following way. For every tree T we consider an auxiliary tree $\bar{T}$ which is obtained from T by the insertion of a new vertex into every internal edge. There will be two types of internal vertices in $\bar{T}$: the "old" vertices, which coincide with the internal vertices of T, and the "new" ones, which can be thought geometrically as the midpoints of the internal edges of T.

To every tail vertex of $\bar{T}$ we assign the embedding i. To every "old" vertex v we assign m_k where k stands for the number of ingoing edges. To every "new" vertex we assign the homotopy operator H. To the root we assign the projector p. Then moving along the tree down to the root one reads off the map $m_{n,T}$ as the composition of maps assigned to vertices of $\bar{T}$.

An analog of PQ-manifold is an A_∞-algebra with an odd innerproduct. Such an algebra can be considered as a linear space E equipped with such an odd inner product $< .,. >$ that $< a_0, m_k(a_1, ..., a_k) >= \mu(a_0, a_1, ..., a_k)$ is cyclically (graded)symmetric. If we have an A_∞-algebra specified by means of polylinear operators $m_k(a_1, ..., a_k)$ on E then the graded symmetrization of m_k specifies maps defining an L_∞-algebra. An odd invariant inner product on A_∞-algebra can be considered as an invariant inner product on the corresponding L_∞-algebra. For any A_∞ -algebra E we can construct an A_∞-algebra E_N as a tensor product $E \otimes Mat_N$. More precisely a basis of E_N consists of elements $(e_k)^\beta_\alpha$ where $1 \leq k \leq n$, $1 \leq \alpha, \beta \leq N$ and

$$m_k((e_{i_k})^{\beta_1}_{\alpha_1}, ..., (e_{i_k})^{\beta_k}_{\alpha_k}) = m^a_{i_1,...,i_1} \delta^{\alpha_2}_{\beta_1} \delta^{\alpha_3}_{\beta_2} ... \delta^{\alpha_n}_{\beta_{n-1}} \delta^{\alpha_1}_{\beta} \delta^{\alpha}_{\beta_n} (e_a)^\beta_\alpha$$

If the A_∞-algebra algebra E is equipped with an odd invariant inner product the same is true for A_∞-algebra E_N and for corresponding L_∞-algebras LE_N.

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